

# OPERATORS WITH ANALYTIC ORBIT FOR THE TORUS ACTION

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**ABSTRACT.** The class of the bounded operators on  $L^2(\mathbb{T}^n)$  with analytic orbit under the action of  $\mathbb{T}^n$  by conjugation with the translation operators is shown to coincide with the class of those pseudodifferential operators on  $\mathbb{T}^n$  whose discrete symbol  $(a_j)_{j \in \mathbb{Z}^n}$  is *uniformly analytic*, in the sense that there exists  $C > 1$  such that the derivatives of  $a_j$  satisfy  $|\partial^\alpha a_j(x)| \leq C^{1+|\alpha|} \alpha!$  for all  $x \in \mathbb{T}^n$ , all  $j \in \mathbb{Z}^n$  and all  $\alpha \in \mathbb{N}^n$ .

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## INTRODUCTION

Let us consider the unitary representations  $y \mapsto T_y$  and  $\eta \mapsto M_\eta$  of  $\mathbb{R}^n$  on  $L^2(\mathbb{R}^n)$  defined by  $T_y u(x) = u(x - y)$  and  $M_\eta u(x) = e^{ix \cdot \eta} u(x)$ ,  $x \in \mathbb{R}$ . Cordes [3] proved that a bounded linear operator  $A$  on  $L^2(\mathbb{R}^n)$  is such that

$$(1) \quad \mathbb{R}^{2n} \ni (y, \eta) \mapsto M_\eta T_y A T_{-y} M_{-\eta} \in \mathcal{L}(L^2(\mathbb{R}^n))$$

is a smooth function with values in the Banach space of all bounded operators on  $L^2(\mathbb{R}^n)$  if and only if there is a  $p \in C^\infty(\mathbb{R}^{2n})$ , bounded and with all its partial derivatives also bounded, such that, for all smooth and rapidly decreasing  $u$  and all  $x \in \mathbb{R}^n$ , one has

$$(2) \quad Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi, \quad \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-is \cdot \xi} u(s) ds.$$

This class of operators is often denoted by  $OPS_{0,0}^0(\mathbb{R}^n)$ , they are the pseudodifferential operators of order zero with symbols satisfying  $\rho = \delta = 0$  Hörmander-type global estimates. Cordes' result can thus be regarded as the characterization of  $OPS_{0,0}^0(\mathbb{R}^n)$  as the operators with smooth orbit under the canonical (not everywhere strongly continuous) action of the Heisenberg group on the  $C^*$ -algebra  $\mathcal{L}(L^2(\mathbb{R}^n))$ . It implies, in particular, that this class is a spectrally invariant  $*$ -subalgebra of  $\mathcal{L}(L^2(\mathbb{R}^n))$ .

The class  $OPS_{0,0}^0(\mathbb{R}^n)$  is not invariant under diffeomorphisms. Then it does not make sense, in general, to define it on manifolds. But since  $OPS_{0,0}^0(\mathbb{R}^n)$  is invariant under translations, it does make sense to define  $OPS_{0,0}^0(\mathbb{T}^n)$  as the class of operators acting on functions defined on the torus  $\mathbb{T}^n$  which are “locally” (the quotation marks indicate that only the canonical charts of the torus are considered) given by

operators in  $OPS_{0,0}^0(\mathbb{R}^n)$ . It follows from Cordes' result on  $\mathbb{R}^n$  that  $OPS_{0,0}^0(\mathbb{T}^n)$  can be characterized as those bounded operators on  $L^2(\mathbb{T}^n)$  which have smooth orbits when acted by the group  $\mathbb{T}^n$  via conjugation with the translations. This is stated, using the discrete symbol representation of pseudodifferential operators on  $\mathbb{T}^n$ , in [6, Theorem 2] for the case  $n = 1$  and here in our Theorem 2.

The main purpose of this paper is to address the question of which bounded operators on  $L^2(\mathbb{T}^n)$  have real analytic orbits. We prove in Theorem 3 that they consist precisely of the operators in  $OPS_{0,0}^0(\mathbb{T}^n)$  whose discrete symbols, which are sequences  $(a_j)$ , are "uniformly analytic", meaning that each  $a_j$  is analytic and that the coefficients of their Taylor series satisfy estimates uniform in  $j$ . Following [2], we denote by  $\mathfrak{Op}_0(\mathbb{T}^n)$  this class of pseudodifferential operators.

The global representation of pseudodifferential operators on the torus, in which the discrete Fourier transform and discrete symbols replace, respectively, the Fourier transform on  $\mathbb{R}^n$  and localization, goes back to Volevich in the 1970's [1]. A complete treatment of this subject, including analogues of the Fourier transform for noncommutative Lie groups, is given in [11].

It is an easy corollary of our Theorem 3 that  $\mathfrak{Op}_0(\mathbb{T}^n)$  is a spectrally invariant  $*$ -subalgebra of  $\mathcal{L}(L^2(\mathbb{T}^n))$ . Chinni and Cordaro [2] give a more concrete proof that it is a  $*$ -algebra, but we don't know if there is a proof of the spectral invariance directly using local or discrete symbols. It also follows that  $\mathfrak{Op}_0(\mathbb{T}^n)$  is dense in  $OPS_{0,0}^0(\mathbb{T}^n)$ .

Characterizations of classes of pseudodifferential operators as the smooth operators for actions of Lie groups on  $C^*$ -algebras have also been considered in [7, 8, 10, 12], in connection with questions arising from Deformation Quantization and Nonlinear PDEs.

## 1. SMOOTHNESS RESULT

Let  $\mathbb{T}^n$  denote the torus  $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ . For each  $j \in \mathbb{Z}^n$ , let  $e_j \in C^\infty(\mathbb{T}^n)$  be defined by  $e_j(x) = e^{ij \cdot x}$ . We have just denoted, as we often will, by the same letter  $x$  both an element  $x$  of  $\mathbb{R}^n$  and its class  $[x] \in \mathbb{T}^n$ . We equip  $\mathbb{T}^n$  with the measure induced by the Lebesgue measure of  $\mathbb{R}^n$ . If  $g \in L^1([-\pi, \pi]^n)$  and  $f([x]) = g(x)$ , we then have

$$\int_{\mathbb{T}^n} f = \int_{[-\pi, \pi]^n} g(x) dx.$$

Given a linear map  $A : C^\infty(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$ , the sequence  $(a_j)_{j \in \mathbb{Z}^n}$ ,  $a_j = e_{-j}(Ae_j) \in C^\infty(\mathbb{T}^n)$ , is called the *discrete symbol* of  $A$ . Using Fourier series, one may then write

$$(3) \quad Au(x) = \frac{1}{(2\pi)^n} \sum_{j \in \mathbb{Z}^n} a_j(x) e_j(x) \hat{u}_j, \quad \hat{u}_j = \int_{\mathbb{T}^n} e_{-j} u,$$

for all  $u \in C^\infty(\mathbb{T}^n)$  and  $x \in \mathbb{T}^n$ .

We say that  $(a_j)_{j \in \mathbb{Z}^n}$ ,  $a_j \in C^\infty(\mathbb{T}^n)$ , is a *symbol of order*  $m \in \mathbb{R}$  if, for every multiindice  $\alpha \in \mathbb{N}^n$ ,

$$\sup\{(1 + |j|)^{-m} |\partial^\alpha a_j(x)|; j \in \mathbb{Z}^n, x \in \mathbb{T}^n\} < \infty,$$

where  $|j| = (j_1^2 + \dots + j_n^2)^{1/2}$ ,  $j = (j_1, \dots, j_n)$ . For  $n = 1$ , the following theorem is Proposition 1 of [6]. To adapt that proof to the case  $n > 1$ , one uses  $e^{ij \cdot x} = (1 + |j|^2)^{-p} |(1 - \Delta)^p e^{ij \cdot x}|$ ,  $j \in \mathbb{Z}^n$ ,  $p \in \mathbb{N}$ , where  $\Delta$  denotes the Laplace operator. See also [2, Theorem 3.4].

**Theorem 1.** *If  $(a_j)_{j \in \mathbb{Z}^n}$  is a symbol of order  $m$ , then the formula (3) defines a linear operator  $A : C^\infty(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$ . If  $m \leq 0$ , then  $A$  extends to a bounded linear operator  $A : L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n)$ , whose norm satisfies, for any integer  $p > \frac{n}{2}$ ,*

$$(4) \quad \|A\| \leq C_p \sup\{|(1 - \Delta)^p a_j(x)|; j \in \mathbb{Z}^n, x \in \mathbb{T}^n\}, \quad C_p = \sum_{l \in \mathbb{Z}^n} (1 + |l|^2)^{-p}.$$

We then write  $A = Op((a_j)_{j \in \mathbb{Z}^n})$ , or simply  $A = Op(a_j)$ .

For each  $y \in \mathbb{T}^n$ , let  $T_y$  denote the operator, unitary on  $L^2(\mathbb{T}^n)$ , defined by  $(T_y u)(x) = u(x - y)$ . For  $A = Op(a_j)$ , one has  $T_y A T_{-y} = Op((T_y a_j)_{j \in \mathbb{Z}^n})$ . For each  $\alpha \in \mathbb{N}^n$ , let us denote  $A^\alpha = Op((\partial^\alpha a_j)_{j \in \mathbb{Z}^n})$ . It follows from the group property  $T_y T_z = T_{y+z}$  and from the estimate (4) that  $y \mapsto T_y A T_{-y}$  is a smooth function on  $\mathbb{T}^n$  with values in the Banach space  $\mathcal{L}(L^2(\mathbb{T}^n))$  and, moreover,  $\partial_y^\alpha (T_y A T_{-y}) = T_y A^\alpha T_{-y}$ . This proves the *if* statement in the following theorem.

**Theorem 2.** *A bounded operator  $A \in \mathcal{L}(L^2(\mathbb{T}^n))$  is such that the map*

$$(5) \quad \mathbb{T}^n \ni y \mapsto T_y A T_{-y}$$

*is smooth with respect to the norm topology of  $\mathcal{L}(L^2(\mathbb{T}^n))$  if and only if  $A = Op(a_j)$  for some symbol  $(a_j)_{j \in \mathbb{Z}^n}$  of order zero.*

**Proof:** If a bounded operator  $A$  on  $L^2(\mathbb{R}^n)$  is such that the map (1) is smooth, we call it a *Heisenberg smooth* operator. Analogously, if  $A \in \mathcal{L}(L^2(\mathbb{T}^n))$  is such that (5) is smooth, we say that it is *translation smooth*.

For each  $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ , let  $\tilde{U}_\theta \subset \mathbb{R}^n$  denote the open rectangle

$$\tilde{U}_\theta = ]\theta_1 - \pi, \theta_1 + \pi[ \times \dots \times ]\theta_n - \pi, \theta_n + \pi[,$$

let  $U_\theta \subset \mathbb{R}^n / (2\pi\mathbb{Z})^n = \mathbb{T}^n$  denote the set of the equivalence classes of all  $x \in \tilde{U}_\theta$  and let  $\chi_\theta : U_\theta \rightarrow \tilde{U}_\theta$  be the chart that sends a class in  $U_\theta$  to its unique representative in  $\tilde{U}_\theta$ . Let then  $I_\theta : L^2(\mathbb{T}^n) \rightarrow L^2(\tilde{U}_\theta)$  denote the unitary map  $I_\theta u = u \circ \chi_\theta^{-1}$ ,  $E_\theta : L^2(\tilde{U}_\theta) \rightarrow L^2(\mathbb{R}^n)$  denote the extension-by-zero embedding and  $R_\theta : L^2(\mathbb{R}^n) \rightarrow L^2(\tilde{U}_\theta)$  denote the restriction map. Finally, let  $\Xi_\theta : \mathcal{L}(L^2(\mathbb{T}^n)) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n))$  denote the isometric \*-homomorphism  $\Xi_\theta(A) = E_\theta I_\theta A I_\theta^{-1} R_\theta$ .

There exists a finite collection of smooth functions  $\{\phi_1, \dots, \phi_q\}$  (we may choose  $q = 3^n$ ) such that  $\sum_{r=1}^q \phi_r \equiv 1$  and, for each pair of integers  $1 \leq m, p \leq q$ , there is

a  $\theta_{mp} \in \mathbb{R}^n$  such that the union of the supports of  $\phi_m$  and  $\phi_p$  is contained in  $U_{\theta_{mp}}$ . Given a translation smooth operator  $A \in \mathcal{L}(L^2(\mathbb{T}^n))$ , it can thus be written as a finite sum of operators of the form  $\phi(M)A\psi(M)$ , where  $\phi(M)$  and  $\psi(M)$  are the operators of multiplication by, respectively, the smooth functions  $\phi$  and  $\psi$ , whose supports are both contained in  $U_\theta$  for some  $\theta \in \mathbb{R}^n$ . It is then enough to show that such a  $\phi(M)A\psi(M)$  equals  $\text{Op}(a_j)$  for some symbol  $(a_j)_{j \in \mathbb{Z}^n}$  of order zero.

Noting that  $T_y\phi(M)T_{-y} = \phi_y(M)$ , where  $\phi_y = T_y\phi$ , we see that the operators  $\phi(M)$  and  $\psi(M)$  are translation smooth and, hence, that  $\phi(M)A\psi(M)$  is also translation smooth. Next we show that  $\tilde{A} = \Xi_\theta[\phi(M)A\psi(M)]$  is Heisenberg smooth. For that it suffices to show (see [4, Section 8.1]) that both maps

$$(6) \quad \mathbb{R}^n \ni y \mapsto T_y\tilde{A}T_{-y} \quad \text{and} \quad \mathbb{R}^n \ni \eta \mapsto M_\eta\tilde{A}M_{-\eta}$$

are smooth (abusing notation, we are denoting by  $T_y$  translation operators both on  $L^2(\mathbb{T}^n)$  and on  $L^2(\mathbb{R}^n)$ ). It suffices to prove that the partial derivatives exist at  $y = 0$  and at  $\zeta = 0$ , respectively, because  $y \mapsto T_y$  and  $\zeta \mapsto M_\zeta$  are representations.

For  $i = 1, \dots, n$ , let  $A_i$  denote the partial derivative with respect to  $y_i$  at  $y = 0$  of  $y \mapsto T_yAT_{-y}$ ,

$$(7) \quad A_i = \lim_{h \rightarrow 0} \frac{T_{(0, \dots, 0, h, 0, \dots, 0)}\phi(M)A\psi(M)T_{(0, \dots, 0, -h, 0, \dots, 0)} - \phi(M)A\psi(M)}{h}.$$

Since

$$\Xi_\theta[T_y\phi(M)A\psi(M)T_{-y}] = T_y\Xi_\theta[\phi(M)A\psi(M)]T_{-y}$$

for all sufficiently small  $y$ , we conclude from (7) that the derivative with respect to  $y_i$  at  $y = 0$  of  $y \mapsto T_y\tilde{A}T_{-y}$  equals  $\Xi_\theta(A_i)$ .

Let  $\bar{\phi}$  and  $\bar{\psi}$  be smooth functions with support contained in  $U_\theta$  such that, for all sufficiently small  $y$ ,  $\bar{\phi}_y\phi \equiv \phi$  and  $\bar{\psi}_y\psi \equiv \psi$ , and hence

$$\bar{\phi}(M)[T_y\phi(M)A\psi(M)T_{-y}]\bar{\psi}(M) = T_y\phi(M)A\psi(M)T_{-y}.$$

Multiplying both sides of (7) on the left by  $\bar{\phi}(M)$  and on the right by  $\bar{\psi}(M)$ , we then conclude that  $\bar{\phi}(M)A_i\bar{\psi}(M) = A_i$ . The argument we used to find the first order derivatives can next be used to find the second order derivatives and, successively, to prove that  $\mathbb{R}^n \ni y \mapsto T_y\tilde{A}T_{-y}$  is  $C^\infty$ .

Take  $\rho \in C_c^\infty(\tilde{U}_\theta)$  such that  $\rho(M)\tilde{A}\rho(M) = \tilde{A}$ . The second map in (6) is smooth because

$$M_\eta\tilde{A}M_{-\eta} = [M_\eta\rho(M)]\tilde{A}[\rho(M)M_{-\eta}]$$

and the maps  $\eta \mapsto M_\eta\rho(M)$  and  $\eta \mapsto \rho(M)M_{-\eta}$  are smooth.

By Cordes's result [3] (see also [4, Theorem VIII.2.1] and [8, Theorem 1]), there exists  $p \in C^\infty(\mathbb{R}^{2n})$ , bounded and with all its partial derivatives also bounded, such that (2) holds. The discrete symbol  $(a_j)$  of  $\phi(M)A\psi(M)$  can then be expressed in terms of  $p$  by the iterated integral

$$a_j(x) = e_{-j}(x)\phi(x)A(\psi e_j)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(x, \xi) \int_{\tilde{U}_\theta} e^{i(x-s) \cdot (\xi-j)} \rho(s) ds d\xi.$$

Using a standard technique of pseudodifferential calculus (for  $n = 1$ , what we need is the estimate (10) for  $l = 0$  in [6]), it follows that  $(a_j)$  is a zero-order discrete symbol.  $\square$

## 2. ANALYTIC OPERATORS

As in the case of complex valued functions (see, for example, [5, Section II.3] or [13, Section V.1]), a smooth function  $f : \mathbb{T}^n \rightarrow \mathcal{L}(L^2(\mathbb{T}^n))$  is analytic, in the sense that at each point its Taylor series converges to  $f$ , if and only if there is  $C > 1$  such that, for every multiindex  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ ,

$$(8) \quad \sup\{|\partial^\alpha f(x)|; x \in \mathbb{T}^n\} \leq C^{1+|\alpha|} \alpha!$$

(as usual, we denote  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $\alpha! = \alpha_1! \dots \alpha_n!$ ).

We say that  $A \in \mathcal{L}(L^2(\mathbb{T}^n))$  is *translation analytic* if the function  $f(y) = T_y A T_{-y}$ ,  $y \in \mathbb{T}^n$ , is analytic. Being translation smooth, a translation analytic operator is, by Theorem 2, necessarily of the form  $A = \text{Op}(a_j)$ , for some symbol  $(a_j)_{j \in \mathbb{Z}^n}$  of order zero. The following theorem gives a necessary and sufficient condition on  $(a_j)$  for  $A$  to be translation analytic.

For each multiindex  $\beta = (\beta_1, \dots, \beta_n)$ ,  $L^\beta = (1 + \partial_1)^{\beta_1} \dots (1 + \partial_n)^{\beta_n}$  defines an invertible operator  $L^\beta : C^\infty(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$ . Its inverse can be expressed using Fourier series:

$$(9) \quad (L^\beta)^{-1} u(x) = \frac{1}{(2\pi)^n} \sum_{l \in \mathbb{Z}^n} \frac{e^{il \cdot x} \hat{u}_l}{(1 + il_1)^{\beta_1} \dots (1 + il_n)^{\beta_n}}.$$

For each positive integer  $m$ , let us define a seminorm  $\rho_m$  on  $C^\infty(\mathbb{T}^n)$  by

$$\rho_m(a) = \sup\{|\partial^\beta a(x)|; |\beta| \leq m, x \in \mathbb{T}^n\}.$$

**Theorem 3.** *An  $A \in \mathcal{L}(L^2(\mathbb{T}^n))$  is a translation analytic operator if and only if  $A = \text{Op}(a_j)$  for some symbol of order zero  $(a_j)_{j \in \mathbb{Z}^n}$  satisfying, for some constant  $C > 1$  and for every multiindex  $\alpha$ ,*

$$(10) \quad \sup\{|\partial^\alpha a_j(x)|; x \in \mathbb{T}^n, j \in \mathbb{Z}^n\} \leq C^{1+|\alpha|} \alpha!.$$

**Proof:** Suppose first that  $A = \text{Op}(a_j)$  for some  $(a_j)$  satisfying (10) and consider  $f(y) = T_y A T_{-y}$ ,  $y \in \mathbb{T}^n$ . Using the remarks and the notation of the paragraph right after Theorem 1, and by the norm estimate (4), we have, for  $p > \frac{n}{2}$ ,

$$\sup\{|\partial^\alpha f(x)|; x \in \mathbb{T}^n\} \leq \|A^\alpha\| \leq 2^p C_p \sup_{j \in \mathbb{Z}^n} \rho_{2p}(\partial^\alpha a_j).$$

It then follows from (10) that

$$\begin{aligned} \sup\{|\partial^\alpha f(x)|; x \in \mathbb{T}^n\} &\leq 2^p C_p C^{1+2p+|\alpha|} [(\alpha_1 + 2p)! \dots (\alpha_n + 2p)!] \\ &\leq \mu^n 2^p C_p C^{2p+1} (2C)^{|\alpha|} \alpha!, \end{aligned}$$

where  $\mu = \sup_{t \geq 0} 2^{-t} (t + 2p)(t + 2p - 1) \dots (t + 1)$ . This implies  $A$  is translation analytic.

Conversely, let  $A$  be translation analytic. By Theorem 2,  $A = \text{Op}(a_j)$  for some zero order discrete symbol  $(a_j)$ . For each nonzero multiindice  $\beta$ , define:

$$B^\beta = \text{Op}[(L^\beta a_j)_{j \in \mathbb{Z}^n}].$$

Then  $T_y B^\beta T_{-y} = L^\beta f(y)$ , where  $f(y) = T_y A T_{-y}$ ,  $y \in \mathbb{T}^n$ , is analytic, by hypothesis. Since  $C > 1$ , we get from (8):

$$\|B^\beta\| \leq 2^{|\beta|} C^{1+|\beta|} \beta!.$$

On the other hand, the discrete symbol of  $A$  can be recovered from the discrete symbol  $(b_j^\beta)_{j \in \mathbb{Z}^n}$  of  $B^\beta$ ,  $b_j^\beta = e_j(B^\beta e_{-j}) = L^\beta a_j$ , using (9):

$$a_j(x) = \frac{1}{(2\pi)^n} \sum_{l \in \mathbb{Z}^n} \frac{e^{ilx} [e_j(B^\beta e_{-j})]_l^\wedge}{(1 + il_1)^{\beta_1} \dots (1 + il_n)^{\beta_n}}.$$

Given now a multiindice  $\alpha$ , let  $\beta = \alpha + (2, \dots, 2)$ . Using that  $|[e_j(B^\beta e_{-j})]_l^\wedge| \leq (2\pi)^n \|B^\beta\|$  for all  $j$  and  $l$ , we get

$$\begin{aligned} |\partial^\alpha a_j(x)| &= \frac{1}{(2\pi)^n} \left| \sum_{l \in \mathbb{Z}^n} \frac{(il)^\alpha e^{il \cdot x} [e_j(B^\beta e_{-j})]_l^\wedge}{(1 + il_1)^{\beta_1} \dots (1 + il_n)^{\beta_n}} \right| \\ &\leq \|B^\beta\| \sum_{l \in \mathbb{Z}^n} \frac{|l_1|^{\alpha_1} \dots |l_n|^{\alpha_n}}{(1 + l_1^2)^{\beta_1/2} \dots (1 + l_n^2)^{\beta_n/2}} \\ &\leq \left( \sum_{p=-\infty}^{\infty} \frac{1}{1 + p^2} \right)^n 2^{|\alpha|+2n} C^{1+2n+|\alpha|} \beta!. \end{aligned}$$

We may now bound  $\beta!$  by a constant times  $\alpha!$ , similarly as we did in the first part of this proof, which will then prove (10).  $\square$

As in [2], we denote by  $\mathfrak{Op}_0(\mathbb{T}^n)$  the class of all  $A = \text{Op}(a_j)$  with  $(a_j)_{j \in \mathbb{Z}^n}$  satisfying (10).

**Corollary 1.** *The class  $\mathfrak{Op}_0(\mathbb{T}^n)$  is  $^*$ -subalgebra of  $\mathcal{L}(L^2(\mathbb{T}^n))$  which is spectrally invariant, in the sense that, if an  $A \in \mathfrak{Op}_0(\mathbb{T}^n)$  is invertible in  $\mathcal{L}(L^2(\mathbb{T}^n))$ , then its inverse  $A^{-1}$  belongs to  $\mathfrak{Op}_0(\mathbb{T}^n)$ .*

**Proof:** If  $f : \mathbb{T}^n \rightarrow \mathcal{L}(L^2(\mathbb{T}^n))$  and  $g : \mathbb{T}^n \rightarrow \mathcal{L}(L^2(\mathbb{T}^n))$  are real analytic, their pointwise product is analytic. It follows that the product of two translation analytic operators is translation analytic. The map  $y \mapsto f(y)^*$  is also analytic. Hence, the adjoint of a translation analytic operator is translation analytic. Moreover if  $f(y)$  is invertible for every  $y \in \mathbb{T}^n$ , then  $y \mapsto f(y)^{-1}$  is also analytic (because  $T_{-y} = T_y^{-1}$  and a pointwise invertible analytic function has an analytic inverse). Then, if  $A \in \mathcal{L}(L^2(\mathbb{T}^n))$  is invertible and translation analytic, so is  $A^{-1}$ .  $\square$

**Corollary 2.**  *$\mathfrak{Op}_0(\mathbb{T}^n)$  is  $L^2$ -operator-norm dense in  $OPS_{0,0}^0(\mathbb{T}^n)$ .*

**Proof:** It is the content of [9, Theorem 4] that the set of analytic vectors of a strongly continuous representation of a Lie group on a Banach space  $\mathfrak{X}$  is dense in  $\mathfrak{X}$ . We apply that theorem to the representation  $y \mapsto T_y AT_{-y}$  of  $\mathbb{T}^n$  on the Banach space  $\mathfrak{X} = \{A \in \mathcal{L}(L^2(\mathbb{T}^n)); y \mapsto T_y AT_{-y} \text{ is continuous}\}$ . By our Theorem 3, the set of analytic vectors for this representation is equal to  $\mathfrak{Op}_0(\mathbb{T}^n)$ , which is dense in  $\mathfrak{X}$  by Nelson's result. On the other hand,  $\mathfrak{X}$  contains the set of smooth vectors, which is equal to  $OPS_{0,0}^0(\mathbb{T}^n)$  by our Theorem 2.  $\square$

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